1. Find the moment generating function of the RV X whose probability function $P[X=x] = \frac{1}{2^x}$, $x = 1, 2, \ldots \infty$. Hence find the Mean.

<u>Solution:</u>

Given $P(X=x) = \frac{1}{2^x}$, $x = 1, 2, \ldots \infty$

So

WKT $M_X(t) = E[e^{tx}] = \sum e^{tx} P(X=x)$

$$= \sum_{x=1,2}^{\infty} e^{tx} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 + x + x^2 + \cdots\right], \text{ where } x = \frac{e^t}{2}$$

$$= \frac{e^t}{2}\left[1 - x\right]^{-1}$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} = \frac{e^t}{2}\left[\frac{2 - e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{\cancel{2}} \times \frac{\cancel{2}}{2 - e^t}$$

$$\boxed{M_X(t) = \frac{e^t}{2 - e^t}}$$

(i) <u>Mean:</u>

$$E[X] = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} \frac{e^t}{2 - e^t}\right]_{t=0}$$

$$= \left[\frac{(2 - e^t)e^t - e^t(0 - e^t)}{(2 - e^t)^2}\right]_{t=0}$$

$$E(X) = \left[\frac{(2-e^0)e^0 + e^0}{(2-e^0)^2}\right]$$

$$= \left[\frac{(2-1)+1}{(2-1)^2}\right]$$

$$\boxed{E(X) = 2}$$

10. If a random variable X has the moment generating function $M_X(t) = \frac{2}{2-t}$. Determine the Variance of X.

Solution:

Gn $M_X(t) = \frac{2}{2-t}$

Mean:

WKt $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{2}{2-t}\right)\right]_{t=0}$

$$= 2\left(\frac{-1}{(2-t)^2}(-1)\right)_{t=0}$$

$$\boxed{\because d\left(\frac{1}{x}\right) = -\frac{1}{x^2}}$$

$$\therefore E(X) = \frac{2}{2^2}$$

$$\boxed{E(X) = \frac{1}{2}}$$

Variance

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} \frac{2}{(2-t)^2}\right]_{t=0}$$

$$\boxed{d\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}}$$

$$= \left[2\left(\frac{-2}{(2-t)^3}(-1)\right)\right]_{t=0}$$

$$E(x^2) = \frac{4}{8}$$

$$\boxed{E(x^2) = \frac{1}{2}}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4}$$

$$\boxed{V(x) = \frac{1}{4}}$$

11. For the triangular distribution $f(x) = \begin{cases} x & ; 0<x<1 \\ 2-x & ; 1<x<2 \\ 0 & ; \text{otherwise} \end{cases}$

Find the mean, Variance and MGF.

Solution :

(i) Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(x)\,dx + \int_1^2 x(2-x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)_1^2$$

$$E(x) = \left(\frac{1}{3} - 0\right) + \left(\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right)$$

$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = \frac{5-2}{3} = \frac{3}{3}$$

$$\boxed{E(x) = 1}$$

(ii) Variance:

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2(x)\,dx + \int_1^2 x^2(2-x)\,dx$$

$$E(x^2) = \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$E(x^2) = \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_1^2$$

$$= \left(\frac{1}{4} - 0\right) + \left(\frac{2\times 2^3}{3} - \frac{2^4}{4}\right) - \left(\frac{2}{3}(1) - \frac{1}{4}\right)$$

$$E(x^2) = \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$E(x^2) = \frac{-14}{4} + \frac{14}{3} = 14\left[\frac{-1}{4} + \frac{1}{3}\right]$$

$$E(x^2) = 14\left[\frac{-3+4}{12}\right]$$

$$E(x^2) = 14\left(\frac{1}{12}\right)$$

$$\boxed{E(x^2) = \frac{7}{6}}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{7}{6} - 1$$

$$\boxed{V(x) = \frac{1}{6}}$$

Moment Generating Function: (MGF)

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^1 e^{tx}(x)\,dx + \int_1^2 e^{tx}(2-x)\,dx$$

$$u = x \qquad v = e^{tx} \qquad u = (2-x)$$

$$u' = 1 \qquad v_1 = \frac{e^{tx}}{t} \qquad u' = -1$$

$$u'' = 0 \qquad v_2 = \frac{e^{tx}}{t^2} \qquad u'' = 0$$

$$M_X(t) = \left(\frac{x.e^{tx}}{t} - \frac{e^{tx}}{t^2}\right)_0^1 + \left((2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right)_1^2$$

$$M_X(t) = \left(\frac{e^t}{t} - \frac{e^t}{t^2} - \left(0 - \frac{1}{t^2}\right)\right) + \left(\left(0 + \frac{e^{2t}}{t^2}\right) - \left(1\frac{e^t}{t} + \frac{e^t}{t^2}\right)\right)$$

$$M_X(t) = \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$M_X(t) = \frac{(e^t - 1)^2}{t^2}$$

12. Let the random variable X has the pdf ~~$f(x) = \frac{1}{2}e^{-\frac{x}{2}}$~~

$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$ Find MGF, Mean and Variance

<u>Solution:</u>

<u>To find MGF:</u>

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\left(\frac{1}{2}-t\right)x}\,dx$$

$$M_X(t) = \frac{1}{2}\left[\frac{e^{-\left(\frac{1}{2}-t\right)x}}{-\left(\frac{1}{2}-t\right)}\right]_0^{\infty}$$

$$M_X(t) = \frac{1}{2}\left[\frac{e^{-\infty}}{-\left(\frac{1}{2}-t\right)} - \frac{e^{0}}{-\left(\frac{1}{2}-t\right)}\right]$$

$$M_X(t) = \frac{1}{2}\left[0 + \frac{1}{\left(\frac{1}{2}-t\right)}\right]$$

$$M_X(t) = \frac{1}{2}\left[\frac{1}{\frac{(1-2t)}{2}}\right]$$

$$\boxed{M_X(t) = \frac{1}{(1-2t)}}$$

Mean:

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} \frac{1}{(1-2t)}\right]_{t=0}$$

$$= \left[\frac{-1}{(1-2t)^2} \times -2\right]_{t=0} \qquad \boxed{d\left(\frac{1}{x}\right) = \frac{-1}{x^2}}$$

$$\boxed{E(X) = 2}$$

Variance:

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} \frac{2}{(1-2t)^2}\right]_{t=0}$$

$$= 2\left[\frac{-2}{(1-2t)^3}(-2)\right]_{t=0}$$

$$\boxed{E(X^2) = 8}$$

$$V(X) = E(X^2) - (E(X))^2 = 8-4$$

$$\boxed{V(X) = 4}$$

13. The density function of a random variable x is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, Mean, Variance and r^{th} moment.

Solution:

To find k:

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^2 kx(2-x)\,dx = 1 \Rightarrow k \int_0^2 (2x - x^2)\,dx = 1$

$\Rightarrow k\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = 1$

$k\left[\left(4 - \frac{8}{3}\right) - 0\right] = 1$

$k\left[\left(\frac{12-8}{3}\right)\right] = 1$

$k\left(\frac{4}{3}\right) = 1$

$$\boxed{k = \frac{3}{4}}$$

$\therefore f(x) = \frac{3}{4}x(2-x)$, $0 \le x \le 2$

(ii) r^{th} moment

$\mu_r' = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$

$= \int_0^2 x^r \frac{3}{4} x(2-x)\,dx$

$= \frac{3}{4}\left[\int_0^2 x^{r+1}(2-x)\,dx\right]$

$= \frac{3}{4} \int_0^2 \left(2x^{r+1} - x^{r+2}\right)dx$

$$E(X^r) = \frac{3}{4}\left(\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right)_0^2$$

$$= \frac{3}{4}\left(\frac{2(2^{r+2})}{r+2} - \frac{2^{r+3}}{r+3} - 0 + 0\right)$$

$$= \frac{3}{4}\left(\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right)$$

$$E(X^r) = \frac{3}{4} \times 2^{r+3}\left(\frac{1}{r+2} - \frac{1}{r+3}\right)$$

$$= \frac{3}{\cancel{4}} \times 2^{r+1} \times \cancel{2}^{\cancel{2}}\left[\frac{r+3-r-2}{(r+2)(r+3)}\right]$$

$$\mu_r' = E(X^r) = 3 \times \frac{2^{r+1}}{(r+2)(r+3)} \quad \text{—— (1)}$$

(iii) <u>Mean:</u> put $r=1$ in (1)

$$\mu_1' = E(X) = 3 \times \frac{2^2}{(1+2)(1+3)}$$

$$\mu_1' = \cancel{3} \times \frac{\cancel{4}}{\cancel{3} \times \cancel{4}}$$

$$\boxed{E(X) = 1}$$

put $r=2$ in (2)

$$\mu_2' = E(X^2) = 3 \times \frac{2^{2+1}}{4 \times 5}$$

$$= 3 \times \frac{\cancel{8}^{2}}{\cancel{4} \times 5}$$

$$\boxed{E(X^2) = \frac{6}{5}}$$

(iv) Variance

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{6}{5} - (1)^2$$

$$V(X) = \frac{6}{5} - 1$$

$$\boxed{V(X) = \frac{1}{5}}$$

14. Find the MGF of the random variable X having the probability density function $f(x) = \begin{cases} 2e^{-2x} , & x > 0 \\ 0 , & x < 0 . \end{cases}$ Also find the first four moments about the origin.

Solution:

$$MGF = M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$M_X(t) = \int_0^{\infty} e^{tx} 2e^{-2x} dx = 2\int_0^{\infty} e^{-(2-t)x} dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$M_X(t) = 2\left[e^{-\infty} - \frac{e^0}{-(2-t)}\right]$$

$$M_X(t) = \frac{2}{(2-t)}$$

WKT $\mu_r' =$ co-efficient of $\frac{t^r}{r!}$ in $M_X(t)$

Consider $M_X(t) = \frac{2}{(2-t)} = \frac{2}{2(1-\frac{t}{2})}$

$$M_X(t) = \left(1-\frac{t}{2}\right)^{-1}$$

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$$M_X(t) = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots\cdots$$

$$= 1 + \frac{1}{2}\left(\frac{t^1}{1!}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{6}{8}\right) + \frac{t^4}{4!}\left(\frac{4!}{16}\right) + \cdots$$

First four moments about the origin

$$\mu_1' = \text{coefficient of } \frac{t^1}{1!} = \frac{1}{2} = E(x)$$

$$\mu_2' = \text{coefficient of } \frac{t^2}{2!} = \frac{1}{2} = E(x^2)$$

$$\mu_3' = \text{Coefficient of } \frac{t^3}{3!} = \frac{6}{8} = \frac{3}{4} = E(x^3)$$

$$\mu_4' = \text{Coefficient of } \frac{t^4}{4!} = \frac{4!}{16} = \frac{1\cdot\cancel{2}\cdot 3\cdot\cancel{4}}{\cancel{16}\,2} =$$

$$\mu_4' = \frac{3}{2} = E(x^4)$$